INTEGRALS INVOLVING GENERALIZED LEGENDRE FUNCTIONS

by

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1. Introduction.

The functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, solutions of the differential equation:

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left\{k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)}\right\}w = 0,$$

introduced in [1] by KUIPERS-MEULENBELD, have been defined for all points of the z-plane in which a cross-cut exists along the real axis from 1 to $-\infty$, and in [2] for the real values of z on the cross-cut for $-1 < z < 1$. These functions will be called *generalized Legendre functions,* whereas $P_k^m(z)$ and $Q_k^m(z)$ which are the special cases for m=n, will be denoted by *associated Legendre functions.*

For the sake of brevity we put:

$$\alpha = k+\tfrac{1}{2}(m+n),\ \beta = k-\tfrac{1}{2}(m-n),\ \gamma = k+\tfrac{1}{2}(m-n),\ \delta = k-\tfrac{1}{2}(m+n).$$

In [3] is given a list of integral transforms whose kernels are associated Legendre functions. In this paper we want to extend this list to integral transforms with $P_k^{m,n}(x)$, $Q_k^{m,n}(x)$, or combinations of these as kernels. In section 2 the result is given without proofs. The evaluation of most of the integrals is carried out by expressing $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ in terms of hypergeometric functions in conjunction with the tabels in [3, 20.2]. A more detailed presentation may be found in [4]. In section 3 some inverse transforms are given, making use of the inversion formulas of BRAAKSMA-MEULENBELD (see [5]). In the integrals, occurring in this section is integrated with respect to the lower parameter k. Finally, in section 4 we mention some useful relations, until now not yet published, between generalized and associated Legendre functions.

2. A. Laplace transforms with generalized Legendre functions.

$$\int_0^\infty e^{-pt}\, t^{c-1}(t+1)^{-\frac{1}{2}n}\, P_k^{m,n}(1+2t)dt = \frac{2^{\frac{1}{2}(n-m)}\, p^{\frac{1}{2}m-c}}{\Gamma(\beta+1)\Gamma(-\gamma)}\, E(\beta+1, -\gamma, c-\tfrac{1}{2}m: 1-m: p) \qquad (1)$$

$\text{Re } c > \tfrac{1}{2}\text{Re } m,\ \text{Re } p > 0.$

For m=n:

$$\int_0^\infty e^{-pt}\, t^{c-1}(t+1)^{-\frac{1}{2}m}\, P_k^m(1+2t)dt = -\frac{1}{\pi}\, p^{\frac{1}{2}m-c}\sin\pi k\ E(k+1, -k, c-\tfrac{1}{2}m: 1-m: p) \qquad (2)$$

$\text{Re } c > \tfrac{1}{2}\text{Re } m,\ \text{Re } p > 0.$

For $c=1-\frac{1}{2}m$ in (2):

$$\int_0^\infty e^{-pt}\,[t(1+t)]^{-\frac{1}{2}m}\, P_k^m(1+2t)dt = \pi^{-\frac{1}{2}}\, p^{m-\frac{1}{2}}\, e^{\frac{1}{2}p}\, K_{k+\frac{1}{2}}(\tfrac{1}{2}p)$$

$\text{Re } m < 1,\ \text{Re } p > 0.$

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This transform has already been found in [3, 4.13(1)].
For m=0 in (2):

$$\int_0^\infty e^{-pt}\, t^{c-1}\, P_k(1+2t)dt = -\frac{1}{\pi}\, p^{-c} \sin \pi k\ E(k+1, -k, c: 1: p)$$

Re c > 0, Re p > 0.

$$\left.\begin{aligned}&\int_0^\infty e^{-pt}\, t^{\lambda+\frac{1}{2}m-1}(t+2)^{\frac{1}{2}n} e^{-\pi i m} Q_k^{m,n}(1+t)dt = \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\delta+1)}\left\{\frac{\sin\beta\pi}{2^{m-n+1}\, p^{\lambda+m} \sin m\pi}\, E(-\beta, \gamma+1, \lambda+m: m+1: 2p) + \right.\\ &\left. - \frac{\sin\alpha\pi}{2^{-n+1}\, p^{\lambda} \sin m\pi}\, E(-\alpha, \delta+1, \lambda: 1-m: 2p)\right\}\end{aligned}\right\} \quad (3)$$

Re λ > 0, Re(λ+m) > 0, Re p > 0.

For m=n (3) is equal to [3, 4.13(16)].

$$\left.\begin{aligned}&\int_0^\infty e^{-pt}\, t^{-\frac{1}{2}m}(t+y)^{-k-1}(t+z)^k (t+y+z)^{\frac{1}{2}m} P_k^{m,n}\left(1 - \frac{2t(t+y+z)}{(t+y)(t+z)}\right)dt = \\ &= 2^{\frac{1}{2}(n-m)} e^{\frac{1}{2}(y+z)p}(yz)^{-\frac{1}{2}}\, p^{-1}\, W_{-k+\frac{1}{2}m-\frac{1}{2},\frac{1}{2}n}(py) W_{k+\frac{1}{2}m+\frac{1}{2},\frac{1}{2}n}(pz)\end{aligned}\right\} \quad (4)$$

Re m < 1, y, z real > 0, Re p > 0.

For y=z=1 and m=n=0 we get [3, 4.13(6)].

$$\left.\begin{aligned}&\int_0^\infty e^{-pt}(1-e^{-t})^{\frac{1}{2}m}(1+e^{-t})^{\sigma} P_k^{-m,-n}(e^t)dt = \\ &= 2^{-p+\frac{1}{2}(m-n)}\, \frac{\Gamma(p-k)\Gamma(p+k+1)}{\Gamma(p+\frac{1}{2}m-\frac{1}{2}n+1)\Gamma(p+\frac{1}{2}m+\frac{1}{2}n+1)}\, . \\ &{}_3F_2(p-k, p+\tfrac{1}{2}m+\sigma+1, p+k+1; p+\tfrac{1}{2}m-\tfrac{1}{2}n+1, p+\tfrac{1}{2}m+\tfrac{1}{2}n+1; \tfrac{1}{2})\end{aligned}\right\} \quad (5)$$

Re m > -1, Re p > -Re k-1, Re p > Re k.

For m=n and $\sigma=\frac{1}{2}n$ we get [3, 4.13(11)].

$$\left.\begin{aligned}&\int_0^\infty e^{-pt}(e^t-1)^{\frac{1}{2}m}\left(\frac{\rho e^t}{\rho-2}-1\right)^{\sigma} P_k^{-m,-n}(\rho e^t-\rho+1)dt = \\ &= \rho^{p-\frac{1}{2}m}(\rho-2)^{-\sigma}\, \frac{\Gamma(p-\sigma-k-\frac{1}{2}m)\Gamma(p-\sigma+k-\frac{1}{2}m+1)}{\Gamma(p-\sigma-\frac{1}{2}n+1)\Gamma(p-\sigma+\frac{1}{2}n+1)}\, . \\ &{}_3F_2(p-\sigma-k-\tfrac{1}{2}m, p+1, p-\sigma+k+1-\tfrac{1}{2}m; p-\sigma-\tfrac{1}{2}n+1, p-\sigma+\tfrac{1}{2}n+1; \tfrac{2-\rho}{2})\end{aligned}\right\} \quad (6)$$

Re ρ > 0, Re m > -1, Re p > Re($\frac{1}{2}$m+σ-k)-1, Re p > Re($\frac{1}{2}$m+σ+k).

For m=n and $\sigma=\frac{1}{2}n$ we get [3, 4.13(12)].

$$\int_0^\infty e^{-pt}(1-e^{-t})^{\frac{1}{2}n}P_k^{m,n}(1-2e^{-t})dt = 2^{\frac{1}{2}(n-m)}\frac{\Gamma(p-\frac{1}{2}m)\Gamma(-p-k-\frac{1}{2}n)\Gamma(-\delta)}{\Gamma(-p-\frac{1}{2}m+1)\Gamma(p-k+\frac{1}{2}n)\Gamma(-\alpha)} \quad (7)$$

Re $n > -1$, Re $p > \frac{1}{2}$Re m.

$$\int_0^\infty e^{-pt}(1-e^{-t})^{-\frac{1}{2}m}P_k^{m,n}(2e^{-t}-1)dt = 2^{\frac{1}{2}(n-m)}\frac{\Gamma(p+\frac{1}{2}n)\Gamma(p-m-\frac{1}{2}n)}{\Gamma(p-k-\frac{1}{2}m)\Gamma(p+k-\frac{1}{2}m+1)} \quad (8)$$

Re $m < 1$, Re $p > \frac{1}{2}|$Re $n|$.

B. Integral transforms with generalized Legendre functions.

$$\int_0^1 x^p(1-x)^q(1-zx)^{-\frac{1}{2}n}P_k^{m,n}(1-2xz)dx =$$

$$= 2^{\frac{1}{2}(n-m)}z^{-\frac{1}{2}m}\frac{\Gamma(p-\frac{1}{2}m+1)\Gamma(q+1)}{\Gamma(1-m)\Gamma(p+q-\frac{1}{2}m+2)}\cdot \quad (9)$$

$${}_3F_2(\beta+1,-\gamma,p-\tfrac{1}{2}m+1;1-m,p+q-\tfrac{1}{2}m+2;z)$$

Re $p > \frac{1}{2}$Re $m-1$, Re $q > -1$, z real, $|z| < 1$.

$$\int_{-1}^{+1}(1-x^2)^{-\frac{1}{2}}P_k^{m,n}(x)dx = \frac{\pi^2 2^{\frac{1}{2}(m+n)}}{\cos\frac{1}{2}n\pi}\frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}-\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\beta+1)\Gamma(\frac{1}{2}\delta+1)} \quad (10)$$

Re $m < 1$, $|$Re $n| < 1$.

$$\int_{-1}^{+1}(1-x)^p(1+x)^{k-p-1}P_k^{m,n}(x)dx = 2^\beta\frac{\Gamma(k-p+\frac{1}{2}n)\Gamma(k-p-\frac{1}{2}n)\Gamma(p-\frac{1}{2}m+1)}{\Gamma(\beta+1)\Gamma(\delta+1)\Gamma(-p-\frac{1}{2}m)} \quad (11)$$

Re$(p-\frac{1}{2}m) > -1$, Re$(k-p) > \frac{1}{2}|$Re $n|$.

$$\int_{-1}^{+1}(1-x)^p(1+x)^qP_k^{m,n}(x)dx =$$

$$= \frac{\Gamma(q+\frac{1}{2}n+1)\Gamma(p-\frac{1}{2}m+1)}{\Gamma(1-m)\Gamma(p+q-\frac{1}{2}(m-n)+2)}2^{p+q-\frac{1}{2}(m-n)+1}\cdot \quad (12)$$

$${}_3F_2(\beta+1,-\gamma,p-\tfrac{1}{2}m+1;1-m,p+q-\tfrac{1}{2}(m-n)+2;1).$$

Re$(p-\frac{1}{2}m) > -1$, Re $q+1 > \frac{1}{2}|$Re $n|$.

$$\int_{-1}^{+1}(1-x)^{-\frac{1}{2}m}(1+x)^q(u+x)^{-\sigma}P_k^{m,n}(x)dx =$$

$$= 2^{q+\frac{1}{2}n-m+1}\frac{\Gamma(q+\frac{1}{2}n+1)\Gamma(q-\frac{1}{2}n+1)}{\Gamma(q-k-\frac{1}{2}m+1)\Gamma(q+k-\frac{1}{2}m+2)}(u-1)^{-\sigma}\cdot \quad (13)$$

$${}_3F_2(q+\tfrac{1}{2}n+1,\sigma,q-\tfrac{1}{2}n+1;q-k-\tfrac{1}{2}m+1,q+k-\tfrac{1}{2}m+2;\frac{2}{1-u})$$

Re $m < 1$, Re $q+1 > \frac{1}{2}|$Re $n|$, u not lying on the cut $(-1,+1)$.

$$\int_{-1}^{+1}(1-x)^{-\frac{1}{2}m}(1+x)^q e^{-ux}P_k^{m,n}(x)dx =$$

$$= 2^{q+\frac{1}{2}n-m+1}\frac{\Gamma(q+\frac{1}{2}n+1)\Gamma(q-\frac{1}{2}n+1)}{\Gamma(q-k-\frac{1}{2}m+1)\Gamma(q+k-\frac{1}{2}m+2)}\cdot e^u\,.\tag{14}$$

$${}_2F_2(q+\tfrac{1}{2}n+1, q-\tfrac{1}{2}n+1;\ q-k-\tfrac{1}{2}m+1, q+k-\tfrac{1}{2}m+2;\ -2u)$$

$\mathrm{Re}\ m < 1$, $\mathrm{Re}\ q+1 > \frac{1}{2}|\mathrm{Re}\ n|$.

$$\int_1^\infty (x-1)^{-\frac{1}{2}m}(x+1)^{-p}P_k^{m,n}(x)dx = 2^{1-m+\frac{1}{2}n-p}\frac{\Gamma(p+k+\frac{1}{2}m)\Gamma(p-k+\frac{1}{2}m-1)}{\Gamma(p-\frac{1}{2}n)\Gamma(p+\frac{1}{2}n)}\tag{15}$$

$\mathrm{Re}\ m < 1$, $\mathrm{Re}(p+k+\frac{1}{2}m) > 0$, $\mathrm{Re}(p-k+\frac{1}{2}m-1) > 0$.

$$\int_1^\infty (x-1)^{-\frac{1}{2}m}(x+1)^{-\frac{1}{2}n}(x+t)^{-2\ell-1}P_k^{m,n}(x)dx =$$

$$= (1+t)^{-\ell-\frac{1}{2}m}(1-t)^{-\ell-\frac{1}{2}n}\frac{\Gamma(\alpha+2\ell+1)\Gamma(-\delta+2\ell)}{\Gamma(2\ell+1)}P_k^{-n-2\ell,-m-2\ell}(t)\tag{16}$$

$\mathrm{Re}\ m < 1$, $|\mathrm{Re}(2k+1)| < \mathrm{Re}(m+n+4\ell+1)$, $|t| < 1$.

$$\int_1^\infty e^{-ax}(x-1)^{-\frac{1}{2}m}(x+1)^{-\frac{1}{2}n}P_k^{m,n}(x)dx = 2^{\frac{1}{2}(n-m)}a^{\frac{1}{2}(m+n)-1}W_{\frac{1}{2}(m-n),k+\frac{1}{2}}(2a)\tag{17}$$

$\mathrm{Re}\ m < 1$, $\mathrm{Re}\ a > 0$.

$$\int_1^\infty e^{-ax}(x-1)^{c-1}(x+1)^{-\frac{1}{2}n}P_k^{m,n}(x)dx =$$

$$= \frac{a^{\frac{1}{2}m-c}e^{-a}}{\Gamma(\beta+1)\Gamma(-\gamma)}E(\beta+1, -\gamma, c-\tfrac{1}{2}m: 1-m: 2a)\tag{18}$$

$\mathrm{Re}\ c > \frac{1}{2}\mathrm{Re}\ m$, $\mathrm{Re}\ a > 0$.

$$\int_1^\infty (x-1)^{-\frac{1}{2}m}(x+1)^{-n}e^{xt}K_{\frac{1}{2}n}[(1+x)t]\,P_k^{m,n}(x)dx =$$

$$= 2^{-\frac{1}{2}n-1}e^t\pi^{-\frac{1}{2}}\cos\tfrac{1}{2}n\pi\ \Gamma(-k+\tfrac{1}{2}m-\tfrac{1}{2})\Gamma(k+\tfrac{1}{2}m+\tfrac{1}{2})t^{\frac{1}{2}m-1}W_{\frac{1}{2}-\frac{1}{2}m,-k-\frac{1}{2}}(4t)\tag{19}$$

$\mathrm{Re}\ m < 1$, $\mathrm{Re}(-k+\frac{1}{2}m-\frac{1}{2}) > 0$, $\mathrm{Re}(k+\frac{1}{2}m+\frac{1}{2}) > 0$, $|\arg t| < \frac{3\pi}{2}$.

$$\int_1^\infty e^{-\pi i m}Q_k^{m,n}(x)dx =$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(2k+2)}\ \frac{\Gamma(1-\frac{1}{2}m)\Gamma(k)}{\Gamma(k-\frac{1}{2}m+1)}\ 2^{-\frac{1}{2}m+\frac{1}{2}n}\tag{20}$$

$${}_3F_2(\beta+1, \delta+1, k;\ 2k+2, k-\tfrac{1}{2}m+2;\ 1)$$

$\mathrm{Re}\ k > 0$, $|\mathrm{Re}\ m| < 2$.

$$\int_1^\infty (x^2-1)^{-\frac{1}{2}} e^{-\pi i m} Q_k^{m,n}(x)dx =$$

$$= 2^{-2k-\frac{1}{2}m+\frac{1}{2}n} \frac{\pi^2}{\cos\frac{1}{2}m\pi} \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\frac{1}{2}\alpha+1)\Gamma(\frac{1}{2}\beta+1)\Gamma(\frac{1}{2}\gamma+1)\Gamma(\frac{1}{2}\delta+1)} \tag{21}$$

$\mathrm{Re}\ k > -1,\ |\mathrm{Re}\ m| < 1.$

$$\int_1^\infty (x-1)^p(x+1)^{-p-k-2} e^{-\pi i m} Q_k^{m,n}(x)dx =$$

$$= 2^{-k-\frac{1}{2}m+\frac{1}{2}n-2} \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(p-\frac{1}{2}m+1)\Gamma(p+\frac{1}{2}m+1)}{\Gamma(k+p-\frac{1}{2}n+2)\Gamma(k+p+\frac{1}{2}n+2)} \tag{22}$$

$\mathrm{Re}\ k > -1,\ \mathrm{Re}\ p > \frac{1}{2}|\mathrm{Re}\ m| -1.$

$$\int_1^\infty (x-1)^p(x+1)^q e^{-\pi i m} Q_k^{m,n}(x)dx =$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(2k+2)} \frac{\Gamma(p-\frac{1}{2}m+1)\Gamma(-p-q+k)}{\Gamma(k-q-\frac{1}{2}m+1)} 2^{p+q-\frac{1}{2}m+\frac{1}{2}n} \tag{23}$$

$$\cdot\ {}_3F_2(\beta+1, \delta+1, k-p-q; 2k+2, k-q-\tfrac{1}{2}m+1; 1)$$

$\frac{1}{2}|\mathrm{Re}\ m| -1 < \mathrm{Re}\ p < \mathrm{Re}\ (-q+k).$

$$\int_1^\infty P_k^{m,n}(x)\, Q_\ell^{m,n}(x)dx =$$

$$= \frac{e^{\pi i m}\, 2^{-m+n}}{(\ell-k)(\ell+k+1)} \frac{\Gamma(\ell+\frac{1}{2}(m+n)+1)\Gamma(\ell+\frac{1}{2}(m-n)+1)}{\Gamma(\ell-\frac{1}{2}(m+n)+1)\Gamma(\ell-\frac{1}{2}(m-n)+1)} \tag{24}$$

$\mathrm{Re}(\ell-k) > 0,\ \mathrm{Re}(\ell+k) > -1,\ \mathrm{Re}\ m < 1.$

One of the theorems of BRAAKSMA-MEULENBELD [5] is:

Let k_1 be a real number with $k_1 > \frac{1}{2}\mathrm{Re}\ m + \frac{1}{2}|\mathrm{Re}\ n| - 1$, and $\varphi(t)$ a function such that for all $a > 1$:

$\varphi(t)\,(t-1)^{-\frac{1}{4}-\frac{1}{2}|\mathrm{Re}\ m|} \in L(1,a)$ *if* $\mathrm{Re}\ m \neq 0$,

$\varphi(t)\,(t-1)^{-\frac{1}{4}} \log(t-1) \in L(1,a)$ *if* $\mathrm{Re}\ m = 0$,

$\varphi(t)t^{-1-k_1} \in L(a,\infty)$.

Let further $\varphi(t)$ be a function of bounded variation in a neighbourhood of $t=x(x > 1)$. Then $\varphi(t)$ satisfies the relations:

$$\frac{1}{2\pi i} \int_{k_1-i\infty}^{k_1+i\infty} dk(2k+1)P_k^{m,n}(x) \int_1^\infty \varphi(t)e^{\pi i m}Q_k^{-m,-n}(t)dt = \tfrac{1}{2}\{\varphi(x-0)+\varphi(x+0)\}, \tag{25}$$

and

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty} dk(2k+1)e^{\pi i m}Q_k^{-m,-n}(x)\int_1^\infty \varphi(t)P_k^{m,n}(t)dt = \tfrac{1}{2}\{\varphi(x-0)+\varphi(x+0)\}. \quad (26)$$

If we apply (25) to (24), we find after some simplifications:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}\left(\frac{1}{\ell-k}-\frac{1}{\ell+k+1}\right)\frac{\Gamma(k+\frac{1}{2}(m+n)+1)\Gamma(k+\frac{1}{2}(m-n)+1)}{\Gamma(k-\frac{1}{2}(m+n)+1)\Gamma(k-\frac{1}{2}(m-n)+1)}P_k^{-m,-n}(x)dk$$

$$= 2^{m-n}P_\ell^{m,n}(x) \quad (27)$$

$k_1 > -\frac{1}{2}\text{Re } m + \frac{1}{2}|\text{Re } n| - 1,\ 2k_1 + 1 > |2\text{Re } \ell + 1|,\ |\text{Re } m| < \frac{3}{4},\ x > 1.$

Application of (26) yields:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}\left(\frac{1}{\ell-k}-\frac{1}{\ell+k+1}\right)Q_k^{m,n}(x)dk = Q_\ell^{m,n}(x) \quad (28)$$

$k_1 > \frac{1}{2}\text{Re } m + \frac{1}{2}|\text{Re } n| - 1,\ 2\text{Re } \ell + 1 > |2k_1 + 1|,\ |\text{Re } m| < \frac{3}{4},\ x > 1.$

This result can be found in a direct way by applying Cauchy's integral formula, since the asymptotic behaviour of $Q_k^{m,n}(x)$ as $k \to \infty$ on $|\arg k| \leq \pi - \eta$ with $0 < \eta < \pi$ is given by:

$$Q_k^{m,n}(x) = k^{m-\frac{1}{2}}(x+\sqrt{x^2-1})^{-k}O(1)\ (x > 1).$$

R e m a r k. The formula (27) may be considered as a continuous analogue of the classical Dougall series expansions.

By applying (26) to (15) we obtain:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\,(p+k+\tfrac{1}{2}m)\Gamma(p-k+\tfrac{1}{2}m-1)e^{\pi i m}Q_k^{-m,-n}(x)dk =$$

$$= 2^{-1+m-\frac{1}{2}n+p}\,\Gamma(p-\tfrac{1}{2}n)\Gamma(p+\tfrac{1}{2}n)(x-1)^{-\frac{1}{2}m}(x+1)^{-p} \quad (29)$$

$k_1 > \frac{1}{2}\text{Re } m + \frac{1}{2}|\text{Re } n| - 1,\ \text{Re}(p+\frac{1}{2}m) - \frac{1}{2} > |k_1+\frac{1}{2}|,\ \text{Re } m < \frac{3}{4},\ x > 1.$

Application of (26) to (17) yields:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)W_{\frac{1}{2}(n-m),k+\frac{1}{2}}(2a)e^{-\pi i m}Q_k^{m,n}(x)dk =$$

$$= 2^{\frac{1}{2}(n-m)}a^{1+\frac{1}{2}(m+n)}e^{-ax}(x-1)^{\frac{1}{2}m}(x+1)^{\frac{1}{2}n} \quad (30)$$

$k_1 > -\frac{1}{2}\text{Re } m + \frac{1}{2}|\text{Re } n| - 1,\ \text{Re } a > 0,\ \text{Re } m > -\frac{3}{4},\ x > 1.$

For m=n (30) is transformed into [5, (8.27)].

Some useful relations between the generalized and the associated Legendre functions are the following:

$$P_k^{m,\frac{1}{2}}(2z^2-1) = 2^{-\frac{3}{2}m+\frac{1}{4}} z^{-\frac{1}{2}} P^m_{2k+\frac{1}{2}}(z). \tag{31}$$

$$P_{-\frac{1}{4}}^{m,n}\left(\frac{2}{z^2}-1\right) = 2^{-\frac{3m}{2}+\frac{n}{2}} z^{\frac{1}{2}} e^{\pm\frac{1}{2}\pi i m} P^m_{n-\frac{1}{2}}(z) \tag{32}$$

(the upper or lower sign according as $\operatorname{Im} z \gtrless 0$).

$$Q_k^{m,\frac{1}{2}}(2z^2-1) = 2^{-\frac{3}{2}m+\frac{1}{4}} z^{-\frac{1}{2}} Q^m_{2k+\frac{1}{2}}(z). \tag{33}$$

$$P_k^{m,\frac{1}{2}}\left(\frac{z^2+1}{z^2-1}\right) = -\frac{2^{-\frac{3}{2}m+\frac{3}{4}} e^{2\pi i k} z^{-\frac{1}{2}}(z^2-1)^{\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(-m-2k-\frac{1}{2})} Q^{-2k-1}_{-\frac{1}{2}-m}(z). \tag{34}$$

Formulas (31) and (33) hold for Re $z \neq 0$, z not on the cross-cut, and for $-1 < z < 1$, $z \neq 0$. Formula (32) holds for Re $z \neq 0$, z not on the cross-cut, and for $-1 < z < 1$, $z \neq 0$ after omitting the factor $e^{\pm\frac{1}{2}\pi i m}$. Formula (34) holds for z not on the cross-cut.

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